RECOGNISING ACHIEVEMENT

ADVANCED GCE **4727/01**

MATHEMATICS

Further Pure Mathematics 3

FRIDAY 6 JUNE 2008 Afternoon

Time: 1 hour 30 minutes

Additional materials: Answer Booklet (8 pages)
List of Formulae (MF1)

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- **You are reminded of the need for clear presentation in your answers.**

This document consists of **4** printed pages.

1 **(a)** A cyclic multiplicative group G has order 12. The identity element of G is e and another element is r, with order 12.

(i) Write down, in terms of e and r, the elements of the subgroup of G which is of order 4. [2]

(ii) Explain briefly why there is no proper subgroup of G in which two of the elements are e and r. [1]

(b) A group H has order mnp, where m, n and p are prime. State the possible orders of proper subgroups of H. [2]

2 Find the acute angle between the line with equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{k} + t(\mathbf{i} + 4\mathbf{j} - \mathbf{k})$ and the plane with equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$. [7]

3 **(i)** Use the substitution $z = x + y$ to show that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + y + 3}{x + y - 1} \qquad \text{(A)}$$

may be written in the form $\dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{2(z+1)}{z-1}$. [3]

(ii) Hence find the general solution of the differential equation (A). [4]

4 **(i)** By expressing $\cos\theta$ in terms of $\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{e}^{-\mathrm{i}\theta}$, show that

$$\cos^5\theta \equiv \tfrac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta).$$ [5]

(ii) Hence solve the equation $\cos 5\theta + 5\cos 3\theta + 9\cos\theta = 0$ for $0 \leqslant \theta \leqslant \pi$. [4]

5 Two lines have equations

$$\frac{x-k}{2} = \frac{y+1}{-5} = \frac{z-1}{-3} \qquad \text{and} \qquad \frac{x-k}{1} = \frac{y+4}{-4} = \frac{z}{-2},$$

where k is a constant.

(i) Show that, for all values of k, the lines intersect, and find their point of intersection in terms of k. [6]

(ii) For the case $k = 1$, find the equation of the plane in which the lines lie, giving your answer in the form $ax + by + cz = d$. [4]

6 The operation $\circ$ on real numbers is defined by $a \circ b = a|b|$.

(i) Show that $\circ$ is not commutative. [2]

(ii) Prove that $\circ$ is associative. [4]

(iii) Determine whether the set of real numbers, under the operation $\circ$, forms a group. [4]

7 The roots of the equation $z^3 - 1 = 0$ are denoted by 1, ω and ω^2.

(i) Sketch an Argand diagram to show these roots. [1]

(ii) Show that $1 + \omega + \omega^2 = 0$. [2]

(iii) Hence evaluate

(a) $(2 + \omega)(2 + \omega^2)$, [2]

(b) $\dfrac{1}{2 + \omega} + \dfrac{1}{2 + \omega^2}$. [2]

(iv) Hence find a cubic equation, with integer coefficients, which has roots 2, $\dfrac{1}{2 + \omega}$ and $\dfrac{1}{2 + \omega^2}$. [4]

8 **(i)** Find the complementary function of the differential equation

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x.$$ [2]

(ii) It is given that $y = p(\ln \sin x) \sin x + qx \cos x$, where p and q are constants, is a particular integral of this differential equation.

(a) Show that $p - 2(p + q) \sin^2 x \equiv 1$. [6]

(b) Deduce the values of p and q. [2]

(iii) Write down the general solution of the differential equation. State the set of values of x, in the interval $0 \leqslant x \leqslant 2\pi$, for which the solution is valid, justifying your answer. [3]

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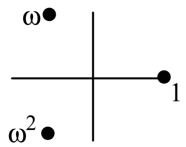
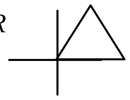
1 (a)(i)	e, r^3, r^6, r^9	M1 A1	 **2**	For stating e, r^m (any $m \geqslant 2$), and 2 other different elements in terms of e and r For all elements correct
(ii)	r generates G	B1	**1**	For this or any statement equivalent to: all elements of G are included in a group with e and r *OR* order of r > order of all possible proper subgroups
(b)	m, n, p, mn, np, pm	B1 B1	 **2**	For any 3 orders correct For all 6 correct and no extras (Ignore 1 and mnp)
		5		
2	METHOD 1			
	$[1, 3, 2] \times [1, 2, -1]$	M1		For attempt to find normal vector, e.g. by finding vector product of correct vectors, or Cartesian equation
	$\mathbf{n} = k[-7, 3, -1]$ *OR* $7x - 3y + z = c\ (= 17)$	A1		For correct vector *OR* LHS of equation
	$\theta = \sin^{-1} \dfrac{\lvert [1, 4, -1] \cdot [-7, 3, -1] \rvert}{\sqrt{1^2 + 4^2 + 1^2}\sqrt{7^2 + 3^2 + 1^2}}$	M1√ M1* M1		For using correct vectors for line and plane f.t. from normal For using scalar product of line and plane vectors For calculating both moduli in denominator
	$\theta = \sin^{-1} \dfrac{6}{\sqrt{18}\sqrt{59}} = 10.6^\circ$	A1√ (*dep)		For scalar product. f.t. from their numerator
	$(10.609...^\circ, 0.18517...)$	A1	**7**	For correct angle
	METHOD 2			
	$[1, 3, 2] \times [1, 2, -1]$	M1		For attempt to find normal vector, e.g. by finding vector product of correct vectors, or Cartesian equation
	$\mathbf{n} = k[-7, 3, -1]$ *OR* $7x - 3y + z = c$	A1		For correct vector *OR* LHS of equation
	$7x - 3y + z = 17$	M1√		For attempting to find RHS of equation f.t. from **n** or LHS of equation
	$d = \dfrac{\lvert 21 - 12 + 2 - 17 \rvert}{\sqrt{7^2 + 3^2 + 1^2}} = \dfrac{6}{\sqrt{59}}$	M1 A1√		For using distance formula from a point on the line, e.g. (3, 4, 2), to the plane For correct distance. f.t. from equation
	$\theta = \sin^{-1} \dfrac{\frac{6}{\sqrt{59}}}{\sqrt{1^2 + 4^2 + 1^2}} = 10.6^\circ$	M1 A1		For using trigonometry For correct angle
	$(10.609...^\circ, 0.18517...)$			
		7		
3 (i)	$\dfrac{dz}{dx} = 1 + \dfrac{dy}{dx}$	M1		For differentiating substitution (seen or implied)
	$\dfrac{dz}{dx} - 1 = \dfrac{z+3}{z-1} \Rightarrow \dfrac{dz}{dx} = \dfrac{2z+2}{z-1} = \dfrac{2(z+1)}{z-1}$	A1 A1	 **3**	For correct equation in z **AEF** For correct simplification to **AG**
(ii)	$\displaystyle\int \frac{z-1}{z+1}\, dz = 2\int dx$	B1		For $\displaystyle\int \frac{z-1}{z+1}(dz)$ and $\int (1)\,(dx)$ seen or implied
	$\Rightarrow \displaystyle\int 1 - \frac{2}{z+1}\, dz$ *OR* $\displaystyle\int 1 - \frac{2}{u}\, du = 2x\ (+c)$	M1		For rearrangement of LHS into integrable form *OR* substitution e.g. $u = z + 1$ or $u = z - 1$
	$\Rightarrow$ $z - 2\ln(z+1)$ *OR* $z + 1 - 2\ln(z+1)$ $= 2x\ (+c)$	A1		For correct integration of LHS as f(z)
	$\Rightarrow -2\ln(x + y + 1) = x - y + c$	A1	**4**	For correct general solution **AEF**

4	(i)	$\cos^5\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^5$	B1		For $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ seen or implied z may be used for $e^{i\theta}$ throughout
		$\cos^5\theta = \frac{1}{32}\left(e^{i\theta}+e^{-i\theta}\right)^5$	M1		For expanding $\left(e^{i\theta}+e^{-i\theta}\right)^5$. At least 3 terms and 2 binomial coefficients required *OR* reasonable attempt at expansion in stages
		$\cos^5\theta = \frac{1}{32}\left(e^{5i\theta}+e^{-5i\theta}+5\left(e^{3i\theta}+e^{-3i\theta}\right)+10\left(e^{i\theta}+e^{-i\theta}\right)\right)$	A1		For correct binomial expansion
		$\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$	M1 A1	5	For grouping terms and using multiple angles For answer obtained correctly **AG**
	(ii)	$\cos\theta = 16\cos^5\theta$	B1		For stating correct equation of degree 5 *OR* $1 = 16\cos^4\theta$ **AEF**
		$\Rightarrow \cos\theta = 0, \quad \cos\theta = \pm\frac{1}{2}$	M1		For obtaining at least one of the values of $\cos\theta$ from $\cos\theta = k\cos^5\theta$ *OR* from $1 = k\cos^4\theta$
		$\Rightarrow \theta = \frac{1}{2}\pi,\ \frac{1}{3}\pi,\ \frac{2}{3}\pi$	A1 A1	4	A1 for any two correct values of θ A1 for the 3rd value and no more in $0 \leqslant \theta \leqslant \pi$ Ignore values outside $0 \leqslant \theta \leqslant \pi$
				9	

5	(i)	METHOD 1 Lines meet where $(x =)\ k+2\lambda = k+\mu$ $(y =)\ -1-5\lambda = -4-4\mu$ $(z =)\ 1-3\lambda = -2\mu$	M1 A1		For using parametric form to find where lines meet For at least 2 correct equations
			M1		For attempting to solve any 2 equations
		$\Rightarrow\ \lambda=-1,\quad \mu=-2$	A1		For correct values of λ and μ
			B1		For attempting a check in 3rd equation *OR* verifying point of intersection is on both lines
		$\Rightarrow\ (k-2, 4, 4)$	A1	6	For correct point of intersection (allow vector)
					SR For finding λ *OR* μ and point of intersection, but no check, award up to M1 A1 M1 A0 B0 A1
		METHOD 2 $d=\dfrac{\lvert[0,3,1].[2,-5,-3]\times[1,-4,-2]\rvert}{\lvert\mathbf{b}\times\mathbf{c}\rvert}$			For using $\mathbf{a}\,.\,\mathbf{b}\times\mathbf{c}$ with appropriate vectors (division by $\lvert\mathbf{b}\times\mathbf{c}\rvert$ is not essential)
		$d=c[0,3,1].[-2,1,-3]=0$ $\Rightarrow$ lines intersect	B1		and showing $d=0$ correctly
		Lines meet where $(x =)\ (k+)\ 2\lambda = (k+)\ \mu$ $(y =)\ -1-5\lambda = -4-4\mu$ $(z =)\ 1-3\lambda = -2\mu$	M1 A1		For using parametric form to find where lines meet For at least 2 correct equations
			M1		For attempting to solve any 2 equations
		$\Rightarrow\ \lambda=-1,\quad \mu=-2$	A1		For correct value of λ *OR* μ
		$\Rightarrow\ (k-2, 4, 4)$	A1		For correct point of intersection (allow vector)
		METHOD 3 e.g. $x-k=\dfrac{2(y+1)}{-5}=\dfrac{y+4}{-4}$	M1		For solving one pair of simultaneous equations
		$\Rightarrow\ y=4$	A1		For correct value of x, y or z
		$\dfrac{z-1}{-3}=\dfrac{y+1}{-5}$	M1		For solving for the third variable
		$x=k-2$ *OR* $z=4$	A1		For correct values of 2 of x, y and z
		$x-k=\dfrac{z}{-2}$ checks with $x=k-2,\ z=4$	B1		For attempting a check in 3rd equation
		$\Rightarrow\ (k-2, 4, 4)$	A1		For correct point of intersection (allow vector)
	(ii)	METHOD 1 $\mathbf{n}=[2,-5,-3]\times[1,-4,-2]$	M1		For finding vector product of 2 directions
		$\mathbf{n}=c[-2,1,-3]$	A1		For correct normal **SR** Following Method 2 for **(i),** award M1 A1√ for **n**, f.t. from their **n**
		$(1,-1,1)$ *OR* $(1,-4,0)$ *OR* $(-1,4,4)$	M1		For substituting a point in LHS
		$\Rightarrow 2x-y+3z=6$	A1	4	For correct equation of plane **AEF cartesian**
		METHOD 2 $\mathbf{r}=[1,-1,1]+\lambda[2,-5,-3]+\mu[1,-4,-2]$	M1		For using vector equation of plane (*OR* $[1,-4,0]$ for **a)**
		$x = 1+2\lambda+\mu$ $y = -1-5\lambda-4\mu$ $z = 1-3\lambda-2\mu$	A1		For writing 3 linear equations
			M1		For eliminating λ and μ
		$\Rightarrow 2x-y+3z=6$	A1		For correct equation of plane **AEF cartesian**
			10		

6 (i)	When a, b have opposite signs,	M1		For considering sign of $a\|b\|$ *OR* $b\|a\|$ in general or in a specific case
	$a\|b\| = \pm ab$, $b\|a\| = \mp ba \Rightarrow a\|b\| \neq b\|a\|$	A1	2	For showing that $a\|b\| \neq b\|a\|$
				Note that $\|x\| = \sqrt{x^2}$ may be used
(ii)	$(a \circ b) \circ c = (a\|b\|) \circ c = a\|b\|\|c\|$ *OR* $a\|bc\|$	M1		For using 3 distinct elements and simplifying $(a \circ b) \circ c$ *OR* $a \circ (b \circ c)$
		A1		For obtaining correct answer
	$a \circ (b \circ c) = a \circ (b\|c\|) = a\|b\|c\|\| = a\|b\|\|c\|$ *OR* $a\|bc\|$	M1		For simplifying the other bracketed expression
		A1	4	For obtaining the same answer
(iii)		B1*		For stating $e = \pm 1$ *OR* no identity
	EITHER $a \circ e = a\|e\| = a \Rightarrow e = \pm 1$	M1		For attempting algebraic justification of +1 and –1 for e
	OR $e \circ a = e\|a\| = a$ $\Rightarrow e = 1$ for $a > 0$, $e = -1$ for $a < 0$	A1		For deducing no (unique) identity
	Not a group	B1 (*dep)	4	For stating not a group
			10	

7 (i)		B1	1	Polar or cartesian values of ω and ω^2 may be used anywhere in this question For showing 3 points in approximately correct positions Allow ω and ω^2 interchanged, or unlabelled
(ii)	*EITHER* $1+\omega+\omega^2$ $=$ sum of roots of cubic $= 0$	M1 A1	2	For result shown by any correct method **AG**
	OR $\omega^3 = 1 \Rightarrow (\omega-1)(\omega^2+\omega+1) = 0$ $\Rightarrow 1+\omega+\omega^2 = 0$ (for $\omega \neq 1$)			
	OR sum of G.P. $1+\omega+\omega^2 = \frac{1-\omega^3}{1-\omega} \left(= \frac{0}{1-\omega}\right) = 0$			
	OR  shown on Argand diagram or explained in terms of vectors			Reference to vectors in part **(i)** diagram may be made
	OR $1+\text{cis}\,\frac{2}{3}\pi+\text{cis}\,\frac{4}{3}\pi = 1+\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i}\right)+\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\text{i}\right) = 0$			
(iii) (a)	$(2+\omega)(2+\omega^2) = 4+2(\omega+\omega^2)+\omega^3$ $= 4-2+1 = 3$	M1 A1	2	For using $1+\omega+\omega^2 = 0$ *OR* values of ω, ω^2 For correct answer
(b)	$\frac{1}{2+\omega}+\frac{1}{2+\omega^2} = \frac{2+(\omega+\omega^2)+2}{3} = 1$	M1 A1√	2	For combining fractions *OR* multiplying top and bottom of 2 fractions by complex conjugates For correct answer f.t. from **(a)**
(iv)	For the cubic $x^3+px^2+qx+r = 0$ METHOD 1			
	$\sum\alpha = 2+1 = 3$ $(\Rightarrow p = -3)$	M1		For calculating two of $\sum\alpha$, $\sum\alpha\beta$, $\alpha\beta\gamma$
	$\sum\alpha\beta = \frac{2}{2+\omega}+\frac{2}{2+\omega^2}+\frac{1}{3} = \frac{7}{3}$ $(= q)$	M1		For calculating all of $\sum\alpha$, $\sum\alpha\beta$, $\alpha\beta\gamma$ *OR* all of p, q, r
	$\alpha\beta\gamma = \frac{2}{3}$ $\left(\Rightarrow\ r = -\frac{2}{3}\right)$	A1		For at least two of $\sum\alpha$, $\sum\alpha\beta$, $\alpha\beta\gamma$ correct (or values of p, q, r)
	$\Rightarrow\ 3x^3-9x^2+7x-2 = 0$	A1	4	For correct equation **CAO**
	METHOD 2 $(x-2)\left(x-\frac{1}{2+\omega}\right)\left(x-\frac{1}{2+\omega^2}\right) = 0$			
	$x^3+\left(-2-\frac{1}{2+\omega}-\frac{1}{2+\omega^2}\right)x^2$ $+\left(\frac{1}{(2+\omega)(2+\omega^2)}+\frac{2}{2+\omega}+\frac{2}{2+\omega^2}\right)x$	M1		For multiplying out LHS in terms of ω or $\text{cis}\,\frac{1}{3}k\pi$
	$-\frac{2}{(2+\omega)(2+\omega^2)} = 0$	M1		For simplifying, using parts **(ii)**, **(iii)** or values of ω
	$\Rightarrow\ x^3-3x^2+\frac{7}{3}x-\frac{2}{3} = 0$	A1		For at least two of p, q, r correct
	$\Rightarrow\ 3x^3-9x^2+7x-2 = 0$	A1		For correct equation **CAO**
			11	

8 (i)	$m^2+1=0 \;\Rightarrow\; m=\pm \mathrm{i}$	M1		For stating and attempting to solve correct auxiliary equation
	$\Rightarrow$ C.F. $(y=)\ C\mathrm{e}^{\mathrm{i}x}+D\mathrm{e}^{-\mathrm{i}x}=A\cos x+B\sin x$	A1	**2**	For correct C.F. (must be in trig form) **SR** If some or all of the working is omitted, award full credit for correct answer
(ii)(a)	$y=p(\ln\sin x)\sin x+qx\cos x$	M1		For attempting to differentiate P.I. (product rule needed at least once)
	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=p\dfrac{\cos x}{\sin x}\sin x+p(\ln\sin x)\cos x+q\cos x-qx\sin x$	A1		For correct (unsimplified) result **AEF**
	$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-p\sin x-p(\ln\sin x)\sin x+\dfrac{p\cos^2 x}{\sin x}-2q\sin x-qx\cos x$	A1		For correct (unsimplified) result **AEF**
	$-p\sin x+\dfrac{p\cos^2 x}{\sin x}-2q\sin x\equiv\dfrac{1}{\sin x}$	M1		For substituting their $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and y into D.E.
		M1		For using $\sin^2 x+\cos^2 x=1$
	$\Rightarrow p-2(p+q)\sin^2 x\equiv 1$	A1	**6**	For simplifying to **AG** ($\equiv$ may be $=$)
(b)		M1		For attempting to find p and q by equating coefficients of constant and $\sin^2 x$ *AND/OR* giving value(s) to x (allow any value for x, including 0)
	$p=1,\ \ q=-1$	A1	**2**	For both values correct
(iii)	G.S. $y=A\cos x+B\sin x+(\ln\sin x)\sin x-x\cos x$	B1√		For correct G.S. f.t. from their C.F. and P.I. with 2 arbitrary constants in C.F. (allow given form of P.I. if p and q have not been found)
	$\operatorname{cosec} x$ undefined at $x=0,\pi,2\pi$ *OR* $\sin x>0$ in $\ln\sin x$	M1		For considering domain of $\operatorname{cosec} x$ *OR* $\sin x\neq 0$ *OR* $\ln\sin x$ term
	$\Rightarrow\ 0<x<\pi$	A1	**3**	For stating correct range **CAO** **SR** Award B1 for correct answer with justification omitted or incorrect
			13	